Recall

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$X, Y \longmapsto [X, Y] \quad \text{Lie bracket}$$

$$[X, Y]f = XYf - YXf$$

Prop.: (Properties of the Lie bracket)

$X, Y, Z \in \mathfrak{X}(M)$

a) bilinearity:- $a, b \in \mathbb{R}$ then

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y]$$

b) Anti-symmetry

$$[X, Y] = -[Y, X]$$

c) Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(future $\mathcal{L}_X Y = [X, Y]$)

d) $\forall f, g \in C^\infty(M)$

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)(X).$$

<u>Proof</u>:- a) and b) from the def^n.

c) Let $f \in C^\infty(M)$ be arbitrary:-

$$[X,[Y,Z]]f + [Y,[Z,X]]f + [Z,[X,Y]]f$$

$$= X[Y,Z]f - [Y,Z]Xf + Y[Z,X]f$$
$$- [Z,X]Yf + Z[X,Y]f - [X,Y]Zf$$

$$= XYZf - XZYf - YZXf + ZYXf$$
$$+ YZXf - YXZf - ZXYf + XZYf$$
$$+ ZXYf - ZYXf - XYZf + YXZf$$

all term cancel pairwise

$= 0.$

d) again a computation. Let $h \in C^\infty(M)$ arbitrary.

$$[fX, gY]h = (fX)(gY)h - (gY)(fX)h$$

$$= (fX)(gYh) - (gY)(fXh)$$

$$= \underbrace{fg(XYh) + f(Yh)(Xg)}_{\text{product rule}} - fg(YXh) - g(Xh)(Yf)$$

$$= fg[X,Y]h + (fXg)(Yh) - (gYf)Xh$$

$= \text{RHS}$. $\boxed{\text{Q}}$

<u>relation of Lie-bracket and F-related v.f.</u>

<u>Prop</u>:- (Naturality of Lie bracket)

Let $F: M \to N$ be a smooth map b/w C^∞ manifolds. and let $X_1, X_2 \in \mathfrak{X}(M)$ which are F-related to $Y_1, Y_2 \in \mathfrak{X}(N)$ respectively.

Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

Proof :- We'll use the criteria that X and Y are F-related $\Leftrightarrow$ $\forall f \in C^\infty(N)$

$$X(f \circ F) = (Yf) \circ F.$$

$$X_1 X_2 (f \circ F) = X_1 (X_2 (f \circ F))$$

$$= X_1 ((Y_2 f) \circ F)$$

X_2 F-related to Y_2

$$= (Y_1 Y_2 f) \circ F$$

X_1 is F-related to Y_1

$$X_2 X_1 (f \circ F) = (Y_2 Y_1 f) \circ F$$

$\Rightarrow$

$$[X_1, X_2](f \circ F) = X_1 X_2 (f \circ F) - X_2 X_1 (f \circ F)$$

$$= (Y_1 Y_2 f) \circ F - (Y_2 Y_1 f) \circ F$$

$$= ([Y_1, Y_2] f) \circ F$$

$\boxed{}$

Prop:-

a) (pushforwards of Lie bracket)

Suppose $F: M \to N$ is a diffeo. and $X_1, X_2 \in \mathfrak{X}(M)$ then

$$F_*[X_1, X_2] = [F_* X_1, F_* X_2]$$

Proof previous prop w/ F being the diffeo.

b) (Bracket of v.f. tangent to submanifolds)

Let M be C^∞ and let S be an embedded submanifold of M. If Y_1 and Y_2 are smooth v.f. on M that are tangent to S then $[Y_1, Y_2]$ is also tangent to S.

Proof Use $F = i : S \hookrightarrow M$ in the previous proposition.

Integral Curves and flows of vector fields

Integral Curves

Suppose M is a C^∞-manifold. If $\gamma : J \to M$ is a smooth curve on M, then the velocity vector

$$\gamma'(t) \in T_{\gamma(t)} M.$$

What can we say about the reverse process?

If $V \in \mathfrak{X}(M)$ can we come up w/ a curve γ s.t the velocity vector = image of V.

Def^n :- Let $V \in \mathfrak{X}(M)$. An integral curve of V is a differentiable curve $\gamma : J \to M$ s.t.

$$\gamma'(t) = V_{\gamma(t)} \quad \forall\ t \in J.$$

If $0 \in J$, then the point $\gamma(0)$ is called the starting point of γ.

e.g.

1) $M = \mathbb{R}^2$ and (x, y) be the coordinates.

Let $V = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$.

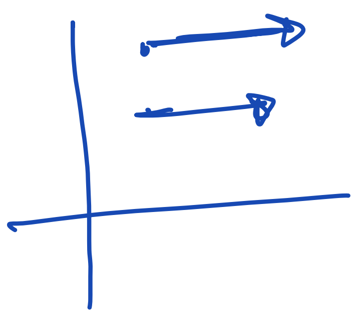
Then $\gamma(t) = (a+t, b)$, a, b constants

is an integral curve for V.

note:- • any $(p, q) \in \mathbb{R}^2$ then $\exists!$ integral curve of V starting from (p, q).

$$\gamma(t) = (p+t, q)$$

• any integral curves of V are either identical or disjoint.



2. $W = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$.

If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a curve, $\gamma(t) = (x(t), y(t))$ were an integral curve for W then

$$\gamma'(t) = \underline{W_{\gamma(t)}}$$

$$\Rightarrow x'(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} + y'(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} = x(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} - y(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)}$$

$\Rightarrow$ we must have

$$x'(t) = -y(t)$$
$$y'(t) = x(t)$$

$$\Rightarrow x(t) = a\cos t - b\sin t \qquad a, b \text{ are constants.}$$
$$y(t) = a\sin t + b\cos t$$

$$\Rightarrow \gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t).$$

is an integral curve of W starting at (a, b).

note:- • given any $(a,b) \in \mathbb{R}^2$ $\exists!$ integral curve γ of W which starts at (a,b).

• images of various integral curves are either identical or disjoint.

Given any $V \in \mathfrak{X}(M)$, finding integral curve of V boils down to solving a system of ordinary differential eqn (ODEs).

i.e. if $U \subseteq M^n$ coordinate chart on M then

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$$

so for γ to be an integral curve of V

we want

$$\gamma^1(t)' = V^1(\gamma^1(t), \dots, \gamma^n(t))$$
$$\gamma^2(t)' = V^2(\gamma^1(t), \dots, \gamma^n(t))$$
$$\vdots$$
$$\gamma^n(t)' = V^n(\gamma^1(t), \dots, \gamma^n(t)).$$

V^i are the component functions of V in U.

<u>Theorem</u> (fundamental theorem for ODEs).

Suppose $U \subseteq \mathbb{R}^n$ is open and let $V : U \to \mathbb{R}^n$ be a C^∞ vector-valued function. Consider the system of ODE

$$\gamma^i(t)' = V^i(\gamma^1(t), \dots, \gamma^n(t)), \quad i = 1, \dots, n$$

$$\gamma^i(t_0) = c^i, \quad i = 1, \dots, n \qquad \text{--- } (1)$$

i.e. $\gamma(t_0) = (c^1, \dots, c^n) \in U \subseteq \mathbb{R}^n$

Then

1) (Existence) $\forall\, t_0 \in \mathbb{R}$ and $x_0 \in U$ $\exists$ an open interval $J_0 \ni t_0$ and an open subset $U_0 \subseteq U$, $x_0 \in U$ s.t. $\forall\, c \in U_0$ $\exists$ a C^1-curve $\gamma : J_0 \to U$ that solves the ODE.

2) (Uniqueness) Any two solutions γ_1 and γ_2 of the ODE (i) are the same on their common domain.

3) The map $\theta : J_0 \times U_0 \to U$ given by $\theta(t, x) = \gamma(t)$ where $\gamma(t)$ is the unique soln w/ $\gamma(t_0) = x$ is a smooth map.

Prop:- Let V be a smooth vector field on a C^∞-manifold M. Then $\forall\, p \in M\ \exists\ \varepsilon > 0$ and a C^∞ curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.

Thus, given any $V \in \mathfrak{X}(M)\ \exists!$ integral curve for some short time which starts from the given point.

Remark :- The fundamental theorem only guarantees the existence of γ on $(-\varepsilon, \varepsilon)$ but not necessarily on whole $\mathbb{R}$.

Summary :- → more properties of the Lie bracket

→ understood what integral curves of a given v.f. are and also saw that they always exist and once we prescribe a starting point they are unique.

In the next in-person lecture :-

- properties of integral curves
- flows of vector fields.

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